Specification Errors in Stress Testing Models

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Abstract

The regulatory and business need to expand the use of stress testing in retail lending has led to a rapid expansion in the types and complexity of models being applied. As these models become more sophisticated and include lifecycle, credit quality, and macroeconomic effects, model specification errors become a common, but rarely identified feature of many of these models. This problem was discovered decades ago in demography with Age-Period-Cohort (APC) models, and we bring those insights to the retail lending context.

Keywords: Age-Period-Cohort Model, Survival Model, Proportional Hazards Model, Dual-time Dynamics, Panel Data Methods, Retail Lending, Stress Testing

1 Introduction

With the introduction of Basel II, stress testing retail loan portfolios has become a high-profile modeling problem. Stress testing is usually interpreted as predicting the future performance of a portfolio under the influence of a severe recession [3]. Many approaches have been proposed [8, 17, 20, 4, 21, 2, 5, 18], but those that are dedicated to retail lending include a quantification of lifecycle, macroeconomic, and credit quality effects [4, 18, 2]. In corporate credit risk models the time effect is calendar time since the lifecycle and vintage effects are not so important there. Although including all three of these factors is essential, it also introduces some important modeling challenges.

The lifecycle captures how consumers interact with the product they purchased as a function of the age of the account. In almost any performance metric considered, lifecycle effects show up as critical to the forecast. Macroeconomic impacts are the most obvious. In practice, changes in employment levels, interest rates, and home price appreciation are observed to be important factors in stress test models [17, 6]. Credit quality refers to variations in the expected performance based upon characteristics at time of origination. Typical factors are bureau scores and demographics, but unobserved factors can also be critical, such as in the US Mortgage Crisis of 2008 [7, 10]. In this it is clear the quality of the borrowers deteriorated markedly between 2005 and 2006. Another example of the unobserved factors is the adverse selection that occurs when a lender undertakes risk based pricing and the quality of those accepting the high interest loans is worse than the lender might expect taking all the observed factors into account.

The modeling may be account-level or vintage-level, but the required model components are the same. A vintage is a group of accounts booked in the same time period, so modeling vintage performance implies modeling the aggregate performance of each vintage rather than modeling the individual accounts within the vintage.

Age Period Cohort models were developed in demography, and studies of those models have revealed certain intrinsic specification errors [19, 12]. This paper will show that the same specification errors occur in retail loan portfolio stress testing, regardless of the modeling framework employed. At the conclusion, we will discuss how these errors may be managed.

2 Indentifying the Specification Errors

Of the many models used for retail loan stress testing, the successful models all have the common features of trying to capture the lifecycle, credit quality, and environmental impacts. These include Dual-time Dynamics [4], Survival or Proportional Hazard Functions [9, 14, 22, 11], Age-Period-Cohort Models, and Panel Data Methods [1, 16]. Differences arise in whether parametric or nonparametric functions are chosen, and whether the modeling is done at the account or vintage level. We will consider each of these situations in turn.

2.1 Nonparametric Models

To investigate the sources of the specification error, let us assume a generic representation of the retail lending stress testing problem to be

$$r(a,t,v) = e^{f_m(a)+f_g(t)+f_Q(v)+\epsilon_{avt}} \tag{1}$$

where r is any rate we wish to model, $f_m(a)$ is the lifecycle function of age of the account (months-on-books), $f_g(t)$ is the measure of the impacts from the environment, and $f_Q(v)$ is the originations quality. For the moment, assume that $f_m(a)$, $f_g(t)$, and $f_Q(v)$ are completely generic, nonparametric functions, as is done in APC and DtD models. ϵ_{avt} is the noise term.

Although not required for estimation, take the log of Equation 2 to obtain

$$\ln r(a,t,v) = f_m(a) + f_g(t) + f_Q(v) + \epsilon_{atv} \tag{2}$$

This agrees with the definition of the age-period-cohort model [13] where the log of the mean of the rate of occurrence of an event is a linear combination of the age, period and cohort terms. Since any nonparametric function can have constant and linear terms, let us represent the three primary functions as

$$\begin{aligned} f_m(a) &= \alpha_0 + \alpha_1 a + f'_m(a) \\ f_g(t) &= \beta_0 + \beta_1 t + f'_g(t) \\ f_Q(v) &= \gamma_0 + \gamma_1 v + f'_Q(v) \end{aligned} \tag{3}$$

Equation 2 then becomes

$$\ln r(a,t,v) = \alpha_0 + \alpha_1 a + f'_m(a) + \beta_0 + \beta_1 t + f'_g(t) + \gamma_0 + \gamma_1 v + f'_Q(v) + \epsilon_{atv} \tag{4}$$

from which it is obvious that we cannot independently estimate three constant terms, α_0, β_0, and γ_0. Therefore, we will let

$$\begin{aligned} \alpha'_0 &= \alpha_0 + \beta_0 + \gamma_0 \\ \beta' &= 0 \\ \gamma' &= 0 \end{aligned} \tag{5}$$

from which we can rewrite Equation 4 as

$$\ln r(a,t,v) = \alpha'_0 + \alpha_1 a + f'_m(a) + \beta_1 t + f'_g(t) + \gamma_1 v + f'_Q(v) + \epsilon_{atv} \tag{6}$$

Conceptually, we have found two specification errors that can be corrected by allowing one of the primary functions to have a constant term and the other two to be measured relative to the first. A logical solution for retail lending is to assume that credit quality and environmental impacts are both measured relative to the lifecycle curve. Whether addressed via substitution as in Equation 6 or via penalty terms in the Likelihood function during estimation, these specification errors are easily enforced.

The third specification error is more challenging. In retail lending we have the additional constraint that the age of an account is completely specified by the date it was booked and the current time,

$$a = t - v \tag{7}$$

We will use this relationship by substituting $v = t - a$ into Equation 6 to obtain

$$\ln r(a,t,v) = \alpha'_0 + \alpha_1 a + f'_m(a) + \beta_1 t + f'_g(t) + \gamma_1 (t-a) + f'_Q(v) + \epsilon_{atv} \tag{8}$$

From Equation 8, it becomes clear that we have one too many linear terms. If we let

$$\begin{aligned} \alpha'_1 &= \alpha_1 - \gamma_1 \\ \beta'_1 &= \beta_1 + \gamma_1 \\ \gamma'_1 &= 0 \end{aligned} \tag{9}$$

then the generic model becomes

$$\ln r(a,t,v) = \alpha_0' + \alpha_1' a + \beta_1' t + f_m'(a) + f_g'(t) + f_Q'(v) + \epsilon_{atv} \tag{10}$$

The conceptual interpretation here is that we can only have two uniquely determined trends among our three nonparametric functions. This is similar to a collinearity problem one might have in ordinary least squares regression. As in regression, several solutions have been proposed for properly estimating these trends. All of the solutions appear to fall into either assuming that one of the functions has zero trend, as done in Equation 8, or to use a method akin to Ridge Regression to apportion the trends across the three functions.

From the calculation of the Hessian matrix, we know that the three specification errors we have found are all that are present. However, the term $f_Q'(v)$ can also be written as $f_Q'(t-a)$. We know that this is a strictly nonlinear function of v, but we are left to wonder if it can be absorbed into $f_m'(a)$ and $f_Q'(v)$.

To investigate the possibility of any nonlinear specification errors, let us note that any function to be evaluated on N points can be represented perfectly by a polynomial of order $N-1$ on those same N points.

$$\begin{aligned} f_m'(a) &= \sum_{i=2}^{N-1} \alpha_i a^i \\ f_g'(t) &= \sum_{i=2}^{N-1} \beta_i t^i \\ f_Q'(v) &= \sum_{i=2}^{N-1} \gamma_i v^i \end{aligned} \tag{11}$$

Thus, we can express Equation 8 as

$$\ln r(a,t,v) = \alpha_0' + \alpha_1' a + \beta_1' t + \sum_{i=2}^{N-1} \alpha_i a^i + \sum_{i=2}^{N-1} \beta_i t^i + \sum_{i=2}^{N-1} \gamma_i v^i + \epsilon_{atv} \tag{12}$$

At this point we again substitute Equation 7 into Equation 12 to obtain

$$\begin{aligned} \ln r(a,t,v) &= \alpha_0' + \sum_{i=1}^{N-1} \left(\alpha_i + (-1)^i \gamma_i\right) a^i + \sum_{i=1}^{N-1} \left(\beta_i + \gamma_i\right) t^i \\ &\quad + \sum_{i=1}^{N-2} \sum_{j=1}^{N-2} (-1)^i \delta_{ij} a^i t^j + \epsilon_{atv} \end{aligned} \tag{13}$$

from which we see that the pure polynomial terms in a or t can be absorbed into $f_m(a)$ or $f_g(t)$ respectively. However, we are left with a collection of cross-terms in a and t which cannot be absorbed into our other functions, but also cannot be expressed back as a modified polynomial in v. The result is that if we want to keep the original structure of three independent functions, we cannot reduce the form beyond that shown in Equation 10.

The ambiguity of trends is critical to the interpretation of stress test models, because it says that we can never be certain whether the trend observed for, say, macroeconomic impacts, is truly from that source or simply a carry-over from trends in credit quality or lifecycle. If a model is used that does not include one of the three functions, $f_m(a)$, $f_g(t)$, or $f_!(v)$, then the structure that would have been present there is instead absorbed by the two remaining functions and the error term, so the interpretation remains ambiguous. Equation 2 has no unique solutions until we impose an assumption about the apportionment of the constant and linear terms.

2.2 Macroeconomic Factor Models

The specification error related to multiple constant terms is obvious and is usually addressed in methods like Survival Analysis and Panel Data Analysis by analyzing mean-zero series. However, as stress test models are being created in retail lending, there is much less recognition of the trend ambiguity when creating factor models. We will consider cases where either the quality or the exogenous functions are replaced with explicit sets of external variables and review the impact on trend degeneracy. We will assume that the lifecycle function $f_M(a)$ remains nonparametric as is the usual approach with Survival Models.

As a first step, let us assume that we are using a Panel Data approach where the lifecycle and quality remain nonparametric functions, i.e. indicator variables are used for each month of the lifecycle curve and each vintage. However, we replace the nonparametric exogenous function with a known economic variable. Unemployment rate, normalized to be mean-zero, would be one such example.

Our model becomes

$$r(a,t,v) = e^{f_m(a)+f_Q(v)+CE(t)+\epsilon_{avt}} \tag{14}$$

where $E(t)$ is our macroeconomic variable with a scaling factor C to calibrate it to retail loan performance. Note that Panel Data analysis and Survival Analysis both analyze account-level data, but by including only a vintage-level indicator function for quality, the individual accounts do not explicitly appear in the model.

Consider the situation where $CE(t)$ is not a complete explanation of performance, but leaves a residual trend. This could occur because of long term changes in fees, credit line increases, collections policies, etc. In fact, it is so common that it should be considered the standard situation. In that case, ϵ_{avt} would not be IID as assumed by the model, but would in fact have a residual trend versus time.

Let us express this residual trend as

$$\epsilon_{avt} = \epsilon_t + \epsilon'_{avt} \tag{15}$$

where ϵ'_{avt} is assumed IID, and ϵ_t captures the net error versus time that is not captured by $CE(t)$. If we split ϵ_t into linear and nonlinear components, then

$$\epsilon_{avt} = et + \epsilon'_t + \epsilon'_{avt} \tag{16}$$

Using the usual substitution from Equation 7, this becomes

$$\epsilon_{avt} = ev - ea + \epsilon'_t + \epsilon'_{avt} \tag{17}$$

Substituting into Equation 14 and using the logarithmic expression gives

$$\ln r(a,t,v) = f_m(a) - ea + f_Q(v) + ev + CE(t) + \epsilon'_t + \epsilon'_{avt} \tag{18}$$

which becomes

$$\begin{aligned} \ln r(a,t,v) &= f'_m(a) + f'_Q(v) + CE(t) + \epsilon'_t + \epsilon'_{avt} && (19) \\ f'_m(a) &= f_m(a) - ea && (20) \\ f'_Q(v) &= f_Q(v) + ev && (21) \end{aligned}$$

The implication is that the residual trend from an imperfect model of calendar time impacts will be absorbed into the maturation and quality functions when a solution is obtained where the residuals have no net mean or trend. Obviously, we can never know if we have a perfect model of calendar time impacts and in fact must always assume that there are unobserved management impacts that would cause exactly this situation. Therefore, the trends in the lifecycle and credit quality functions will depend upon which $E(t)$ is chosen.

This discussion considered a univariate $E(t)$ for simplicity, but this can be changed to a multifactor model, substituting $C\dot{E}(t)$ for $CE(t)$. Making this substitution would have no impact upon the the results shown.

2.3 Incorporating Scores

The next step in our progression is to recast the credit quality function as sensitivity to a known credit risk measure. Because of Equation 7, the net credit risk behavior with vintage is the key factor for this discussion, but practitioners usually express this as an account-level model, so we will adopt an equivalent notation as

$$r(i,a,t,v) = e^{f_m(a) + BS(i,v) + CE(t) + \epsilon_{avt}} \tag{22}$$

This assumes that we have a credit score $S(i)$ which provides a known measure of credit quality for loan i. Each loan is originated at a specific point in time, so it will belong to a unique vintage v. Thus, we can express the quality for account i in vintage v as $S(i,v)$. We know that scores are designed to rank order the accounts by credit risk, but we do not have a precise calibration of score relative to the lifecycle, so we need to introduce a scaling factor B.

If we were certain that the macroeconomic impacts $E(t)$ and credit quality measures $S(i,v)$ were both perfect measures of the trends along those dimensions, then we would have a single, unambiguous solution for our model. The problem is that we can never have that certainty. We discussed earlier why we can never be certain of the calendar-based trends. Measures of quality suffer from similar uncertainties. Breeden, Thomas, and McDonald [7] show that adverse selection was a dominant factor in the US Mortgage Crisis of 2008. Even after normalizing for score, loan-to-value, and product features, measures of credit quality since 2003 show a strong trend with vintage. This is in line with industry reports that credit score did not capture the deterioration in those vintages. [10]

The US Mortgage Crisis is an extreme example, but we know that unpredicted trends in quality occur frequently for both individual institutions and the entire industry as unmeasured changes in consumer attitudes and lenders' policies have significant impacts on loan performance.

Mathematically, the impact of adverse selection is that we must assume that ϵ_{avt} will have potential unobserved trends in both calendar date t and vintage v. As demonstrated previously, some of these unobserved trends will be absorbed into the lifecycle function so that we cannot be certain of the true consumer lifecycle.

The more pernicious problem is that $S(i, v$ and $E(t)$ are never known with certainty. If we allow for any amount of exploration to try to find the best macroeconomic and credit quality factors, we return to a situation where the linear trend component is completely ambiguous.

The conclusion is that because of the interdependence between age, vintage, and time, any model that includes all three of those components will have a model specification error around the linear trend, regardless of whether the model is nonparametric or factor-based.

3 Looking for More

The previous analysis was essentially an algebraic approach to showing specification errors within stress testing models. Although useful for finding errors, the algebraic approach does not provide certainty that all the specification errors were found. To prove that the errors just found reflect all the errors present, we compute employ a Jacobian matrix.

We begin by creating a system of equations sufficient for obtaining an analytical solution to the problem,

$$g_i = x(a, v, t) - f(a, v, t; p) = 0 \tag{23}$$

where x is the observed value, f is the predictive function, and p is the set of parameters. If we have N_p parameters, then we need N_p equations g_i.

Then we take the determinant of the matrix of partial derivatives

$$J = \det \left\{ \begin{array}{ccc} \frac{\partial g_1}{\partial p_1} & \cdots & \frac{\partial g_1}{\partial p_{N_p}} \\ \vdots & & \\ \frac{\partial g_{N_p}}{\partial p_1} & \cdots & \frac{\partial g_{N_p}}{\partial p_{N_p}} \end{array} \right\} \tag{24}$$

By the Inverse Function Theorem, if $J \neq 0$, then the system has a unique solution. When $J = 0$, we can compute the determinant of each of the submatrices obtained by removing one row and one column to identify which parameters are indeterminant.

For a stress testing model, the observation in Equation 5 that we can have only one constant term is fairly obvious and will not be revisited here. However, the discussion of the linear terms is more interesting, so let us consider the case of Equation 6, assuming initially a strictly linear form, meaning that $f'_m(a) = f'_g(t) = f'_Q(v) = 0$.

This gives us

$$g_i = \ln r(a,t,v) - \alpha'_0 + \alpha_1 a + \beta_1 t + \gamma_1 v \tag{25}$$

Here we have four parameters, so we can employ the following specific set of equations

$$\begin{aligned} g_1 &= \ln r(1,1,1) - \alpha'_0 + \alpha_1 + \beta_1 + \gamma_1 \\ g_2 &= \ln r(2,2,1) - \alpha'_0 + 2\alpha_1 + 2\beta_1 + \gamma_1 \\ g_3 &= \ln r(1,2,2) - \alpha'_0 + \alpha_1 + 2\beta_1 + 2\gamma_1 \\ g_4 &= \ln r(2,3,2) - \alpha'_0 + 2\alpha_1 + 3\beta_1 + 2\gamma_1 \end{aligned}$$

This represents two observations each of two vintage time series. Although this should be sufficient to estimate the four parameters, the Jocobian is

$$J = \begin{vmatrix} \frac{\partial g_1}{\partial \alpha'_0} & \frac{\partial g_1}{\partial \alpha_1} & \frac{\partial g_1}{\partial \beta_1} & \frac{\partial g_1}{\partial \gamma_1} \\ \frac{\partial g_2}{\partial \alpha'_0} & \frac{\partial g_2}{\partial \alpha_1} & \frac{\partial g_2}{\partial \beta_1} & \frac{\partial g_2}{\partial \gamma_1} \\ \frac{\partial g_3}{\partial \alpha'_0} & \frac{\partial g_3}{\partial \alpha_1} & \frac{\partial g_3}{\partial \beta_1} & \frac{\partial g_3}{\partial \gamma_1} \\ \frac{\partial g_4}{\partial \alpha'_0} & \frac{\partial g_4}{\partial \alpha_1} & \frac{\partial g_4}{\partial \beta_1} & \frac{\partial g_4}{\partial \gamma_1} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 3 & 2 \end{vmatrix} = 0 \tag{26}$$

Equation 26 shows that we cannot uniquely determine all four parameters. However, if we eliminate any of the three linear terms, α_1, β_1, or γ_1, then $W \neq 0$ and the system is determined. We must fix one of the linear trends in order to obtain a unique solution.

Note that any function evaluated on two points can be expressed linearly, so this result applies to all of the stress test models described so far, whether they use parametric or non-parametric formulations.

To test the nonlinear structure, we adopt the polynomial expression in Equation 12. For any specific polynomial order $N-1$, we can compute the determinant and test if the system has a unique solution. We have computed the results for parabolic, cubic, and quadratic polynomials in a, t, and v and find that they always have unique solutions. This confirms the algebraic approach used in Equation 13 and suggests that we get unique solutions for all nonlinear terms. This has been proven previously in the Age Period Cohort literature.

4 Remedies

No universal solution exists for the model specification error described here. All the solutions in the APC literature are ultimately problem-specific. In the case of retail lending, we need to decide if a specific solution exists.

So far we have analyzed models that include age, time, and vintage (or consumer credit) effects. It is important to consider models that do not include all three effects. At one extreme, we have logistic regression models [15]. In standard implementations, they are used to create credit scores with no dependence on time or age. Clearly such models do not suffer from a model-specification error relative to the trend ambiguity described above, but this is accomplished by implicitly assuming that any trend along those other two dimensions is identically zero. Given the usual practice of creating scores from short periods of time, we are almost guaranteed that this will represent only a fraction of an economic cycle, and thus, intuitively, a strong trend with time should exist. By assuming that trend to be zero, it is necessarily being absorbed into the score.

As a result, we obtain a unique solution for the score, but not one that generalizes out-of-sample. This is one reason credit scores are rebuilt every couple of years. It is an attempt to adjust for the flawed assumption of flat environment and lifecycle. In general, any model that does not include one of these three dimensions will obtain a unique solution, but accomplished by the indefensible assumption that the missing component has a zero trend.

This problem only disappears when very long data sets are analyzed. If a 10- to 20-year data set is analyzed, on that time scale, impacts from the macroeconomic environment should show no net trend, as these series appear to be mean-reverting processes.

Since no universal solution exists for the model specification problem, and a retail-specific solution of zero trend in macroeconomic impacts is only valid for long data sets, we are left to decide problem-by-problem if it is plausible to assume that one of the three components has no net trend, or that the trend can be fixed at a given level. What is clear is that the age effect is strong in consumer credit and can be estimated at the individual loan level using proportional hazard models [2, 18] or at the vintage level with Dual-time Dynamics [4]. An obvious choice of the constant and trend term to use for the age effect is those which are obtained from these methods for a portfolio whose vintages span a long time interval. Estimating the constant and trend in the vintage quality effect seems the most difficult to do as it is caused by both the changes in consumer appetite, in the market conditions, and in the lender's marketing and risk assessment policies.

5 Conclusions

From a modeling perspective, the model specification error described here is easily solved in-sample. The real problem occurs out-of-sample. If we are creating a forecasting or stress testing model, then the forecast period where we need to extrapolate the in-sample macroeconomic impacts may be embedding a trend that has nothing to do with the macroeconomic environment. Of course, we can always extrapolate observed trends into the forecast, but we loose explanatory power. On a short data set, we cannot be certain of the true sensitivity to the macroeconomic environment, because some of that trend may just be bleeding over from an unmodeled trend in credit risk or lifecycle. This has critical implications for the use of stress test models, because any such model built on a short data set cannot definitively be used to model extreme events as is the hope under Basel II. We cannot be certain that the result represents a 99th percentile event or just a 70th percentile event, because we cannot be certain of the sensitivity to macroeconomic factors.

References

[1] Manuel Arellano. *Panel Data Econometrics*. Oxford University Press, 2003.

[2] T. Bellotti and J.N. Crook. Credit scoring with macroeconomic variables using survival analysis. online, 2008.

[3] J. Berkowitz. A coherent framework for stress testing. *Journal of Risk*, 2:1–11, 2000.

[4] Joseph L. Breeden. Modeling data with multiple time dimensions. *Computational Statistics & Data Analysis*, 51:4761 – 4785, May 17 2007.

[5] Joseph L. Breeden. Survey of retail loan portfolio stress testing. In Daniel Rsch and Harald Scheule, editors, *Stress-testing for Financial Institutions - Applications, Regulations and Techniques*. Risk books, 2009.

[6] Joseph L. Breeden and Lyn C. Thomas. The relationship between default and economic cycle for retail portfolios across countries: identifying the drivers of economic downturn. *Journal of Risk Model Validation*, 2(3):11 – 44, 2008.

[7] Joseph L. Breeden, Lyn C. Thomas, and John McDonald III. Stress testing retail loan portfolios with dual-time dynamics. *Journal of Risk Model Validation*, 2(2):43 – 62, 2008.

[8] Committee on the Global Financial System. A survey of stress tests and current practice at major financial institutions. *Available online at http://www.bis.org*, April 2001.

[9] D. R. Cox and D. O. Oakes. *Analysis of Survival Data*. Chapman and Hall, London, 1984.

[10] Y Demyanyk and O van Hemert. Understanding the subprime mortgage crisis. online, 2009.

[11] Bradley Efron. The two-way proportional hazards model. *Journal of the Royal Statistical Society B*, 64:899 – 909, 2002.

[12] Norval D. Glenn. *Cohort Analysis, 2nd Edition*. Sage, London, 2005.

[13] T R Holford. The estimation of age, period and cohort effects for vital rates. *Biometrics*, 39:311–324, 1983.

[14] David W. Hosmer, Jr. and Stanley Lemeshow. *Applied Survival Analysis: Regression Modeling of Time to Event Data.* Wiley Series in Probability and Statistics, New York, 1999.

[15] David W. Hosmer, Jr. and Stanley Lemeshow. *Applied Logistic Regression, 2nd Edition.* Wiley Series in Probability and Statistics, New York, 2000.

[16] Cheng Hsiao. *Analysis of Panel Data.* Cambridge University Press, 2003.

[17] C. Lily and L.P. Hong. *FSAP Stress Testing: Singapore's Experience.* Number 34. August 2004.

[18] M. Malik mand L.C.Thomas. Modelling credit risk of portfolios of consumer loans. online, 2008.

[19] W.M. Mason and S. Fienberg. *Cohort Analysis in Social Research: Beyond the Identification Problem.* Springer, 1985.

[20] R. Perli and W. I. Nayda. Economic and regulatory capital allocation for revolving retail exposures. *Journal of Banking and Finance*, 28:789–809, 2004.

[21] Daniel Rösch and Harald Scheule. Stress-testing credit risk parameters: an application to retail loan portfolios. *Journal of Risk Model Validation*, 1(1):55–75, 2007.

[22] Terry M. Therneau and Patricia M. Grambsch. *Modeling Survival Data: Extending the Cox Model.* Springer-Verlag, New York, 2000.